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Pyramidal tours and multiple objectives

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Abstract In this study, we work on the traveling salesperson problems and bottleneck traveling salesperson problems that have special matrix structures and lead to polynomially solvable cases. We extend the problems to multiple objectives and investigate the properties of the nondominated points. We develop a pseudo-polynomial time algorithm to find a nondominated point for any number of objectives. Finally, we propose an approach to generate all nondominated points for the biobjective case.

Keywords Pyramidal tour · Polynomially solvable · Multiple objectives · TSP · BTSP

1 Introduction

Let $G = (N, E)$ be a graph where N and E denote the sets of nodes and edges, respectively. A traveling salesperson starts a tour from a node, visits all nodes exactly once and returns to the node where the tour is started. Such a tour is called a *Hamiltonian tour*. The node set may stand for the cities and the edge set for the roads directly connecting the cities. For each edge $e \in E$, a weight $w(e)$ is given. This weight may correspond to different objectives, such as duration, cost, distance, risk, etc. associated with traversing the edge. The traveling salesperson problem (TSP) is concerned with finding the Hamiltonian tour with the minimum total weight. In a variant of TSP, the salesperson is not interested in the total distance traveled but in the maximum distance traveled between any two succeeding cities. The problem of finding the Hamiltonian tour whose longest edge is as short as possible is called as *Bottleneck TSP* (*BTSP*). Without loss of generality, we assume that all edge weights are integers.

It is well known that both TSP and BTSP are *NP*-hard in general. However, there are special cases of these problems that are solvable in polynomial time. The special cases of

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TSP and BTSP either have a special distance matrix or a special graph structure. Van der Veen et al. (1991) discuss a solvable case of TSP and propose a proof on the properties of the optimal tour. Van Dal et al. (1993), Oda (2002), and Baki (2006) introduce new solvable cases. Özpeynirci and Köksalan (2009) study multiobjective TSP and BTSP on special graph structures called "Halin graphs". They discuss the complexity of multiobjective TSP and BTSP on these graphs for different objective function combinations. We refer the reader to the surveys of Kabadi (2002), Burkard et al. (1996) for further information on the polynomially solvable cases of the TSP.

Let φ be a Hamiltonian tour on G and let F denote the set of all Hamiltonian tours on G. Let $\varphi(i)$ represent the node succeeding node i in tour φ. A tour can be represented by $\varphi = (1, i_1, i_2, \ldots, i_{n-1})$ where $\varphi(1) = i_1, \varphi(i_1) = i_2, \ldots, \varphi(i_{n-1}) = 1$. Let d be the distance matrix and $d[i, j]$ denote the distance between nodes i and j.

TSP can be stated as:

$$\min_{\phi \in F} \left\{ d(\phi) = \sum_{i=1}^{n} d[i, \phi(i)] \right\} \text{ and}$$

BTSP can be stated as:

$$\min_{\varphi \in F} \left\{ \max_{i=1,\ldots,n} (d\,[i, \varphi\,(i)]) \right\}$$

In a multiobjective TSP and BTSP, there are multiple distance matrices and each tour yields a vector of the objective function values. The problem is to minimize each objective function simultaneously. Since TSP and BTSP are *NP*-Hard even with a single objective, multiobjective versions of these problems are also *NP*-Hard. We refer to Ehrgott and Gandibleux (2002) for a review on multiobjective TSP and BTSP.

In this paper, we study multiobjective TSP and BTSP with special distance matrices. We explore the properties of these problems and develop an exact algorithm to find nondominated points. To the best of our knowledge, this is the first study that addresses the polynomially solvable cases of the multiobjective TSP with special distance matrices.

In the next section, we review a special type of Hamiltonian tours, namely pyramidal tours and explore some of their properties. We further study the pyramidal tours for the multiobjective TSP in Sect. 3. We consider two special cases in Sects. 4 and 5. We make concluding remarks and discuss future research directions in Sect. 6.

2 Pyramidal tours

A tour ρ is pyramidal if starting from node 1, a set of nodes are visited in ascending order up to node n and the remaining nodes are visited in descending order. Formally, tour ρ is called pyramidal if $\rho = (1, i_1, i_2, \ldots, i_k, n, j_1, j_2, \ldots, j_m)$ such that $1 < i_1 < i_2 < \cdots < i_k < n$ and $n > j_1 > j_2 > \ldots > j_m > 1$.

Consider two tours, $\rho = (1, 2, 5, 6, 4, 3)$ and $\phi = (1, 2, 5, 4, 6, 3)$. In Fig. 1, we plot the node numbers in the order they are visited. The plot of tour ρ (Fig. 1a) looks like a pyramid and it has only one peak. Tour ρ is a pyramidal tour. On the other hand, the plot of tour φ (Fig. 1b) has two peaks (nodes 5 and 6) and φ is a non-pyramidal tour.

A complete graph has an edge directly connecting each pair of nodes. If the cost of traversing an edge is independent of the direction of the traverse for all edges (i.e., $d\,[i, j] = d\,[j, i]$ for all (i, j) pairs) then the graph is said to be undirected (symmetric). If $d\,[i, j] \neq d\,[j, i]$ for some (i, j) pair then the graph is said to be directed (asymmetric). We mainly use the

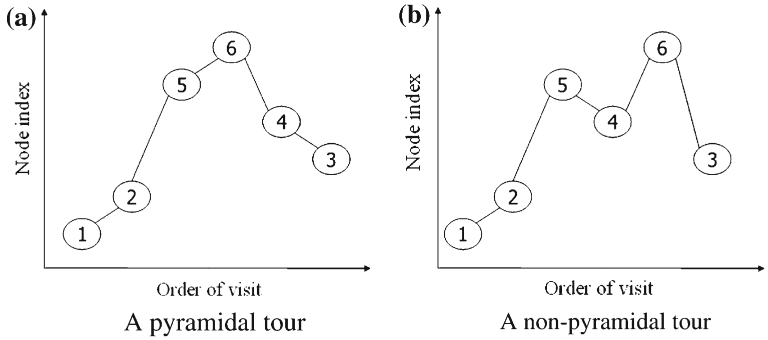

Fig. 1 Plots of the tours

term *edge* for the undirected graphs and *arc* for the directed graphs. Gutin et al. (2002) prove that the number of pyramidal tours is 2^{n-3} in an undirected complete graph and 2^{n-2}in a directed graph. In both cases, the number of pyramidal tours is an exponential function of the number of nodes, n. Let F_{PYR} be the set of all pyramidal tours for a given graph. By definition, $F_{PYR} \subseteq F$ where F is the set of all Hamiltonian tours.

Although the number of pyramidal tours is exponential in n, finding the shortest pyramidal tour for any distance matrix has a complexity of $O(n^2)$ using the dynamic program given in Gilmore et al. (1985). Let $\mathbf{D}_{\mathrm{PYR}}$ denote the family of distance matrices for which a pyramidal tour is optimal. Then, for any matrix in $\mathbf{D}_{\mathrm{PYR}}$, TSP is polynomially solvable.

Tour improvement (*TI*) technique is a proof technique developed by Van der Veen (1994). TI is used to prove that for a class of matrices in $\mathbf{D}_{\mathrm{PYR}}$, the optimal tour is pyramidal. TI starts with an initial tour and iterates by exchanging a set of arcs with others to obtain a new tour and generates a sequence of tours without increasing the tour length. The new tour's length must be at most as large as that of the previous tour in order for this exchange to be a feasible transformation. TI is a framework of feasible transformations that needs to be developed for each class of matrices in $\mathbf{D}_{\mathrm{PYR}}$. A feasible transformation for a class may not be feasible for another.

There are different classes of distance matrices in $\mathbf{D}_{\mathrm{PYR}}$ that have been defined in the literature. There are symmetric and asymmetric matrices in $\mathbf{D}_{\mathrm{PYR}}$. Let $\mathbf{D}_{\mathrm{TI}}$ denote a family of matrix classes. $\mathbf{D}_{\mathrm{TI}}$ includes a distance matrix $\mathbf{D}$, if matrix $\mathbf{D}$ satisfies a property $\mathbf{P}$ such that for any non-pyramidal tour φ, there exists a pyramidal tour $\rho^{(\mathbf{P},\varphi)}$ such that for every matrix satisfying property $\mathbf{P}$, tour $\rho^{(\mathbf{P},\varphi)}$ is at least as good as tour φ. In this case, the existence of this property guarantees transforming any non-pyramidal tour into a pyramidal tour that is at least as good through a set of iterative exchanges of arcs. Burkard et al. (1998) applied TI technique to Monge, Supnick, Demidenko, Kalmanson, Van der Veen matrices and generalized distribution matrices, hence these matrices are in $\mathbf{D}_{\mathrm{TI}}$. By definition $\mathbf{D}_{\mathrm{TI}} \subseteq \mathbf{D}_{\mathrm{PYR}}$.

Note that, for a matrix in $\mathbf{D}_{\mathrm{TI}}$, a non-pyramidal tour φ may also be optimal, giving the same length as the optimal pyramidal tour. A trivial case is a TSP where all arc lengths are equal. TI technique implies that there exists at least one pyramidal tourρ, which can be obtained from φ by applying a sequence of feasible transformations. If φ is optimal, ρ should also be optimal. This is possible if all feasible transformations used to obtain ρ from φ keep the tour length unchanged, i.e. none of the feasible transformations improve the tour length. By definition, feasible transformations cannot increase the tour length. On the other hand, if all possible feasible transformations strictly decrease the tour length, then a non-pyramidal tour cannot be optimal, since for every non-pyramidal tour, there exists at least one pyramidal tour that has a strictly shorter length.

A matrix that is in $\mathbf{D}_{\mathrm{PYR}}$ may not be readily recognizable and may require a renumbering of the nodes to be recognized. There are polynomial time algorithms for recognizing some of these matrices in $\mathbf{D}_{\mathrm{PYR}}$ (see, for example, Burkard and Deineko 2004, and Burkard et al. 1996).

3 The multiobjective TSP

Consider p objective functions corresponding to p distance matrices where each objective function is to be minimized. If all objective functions are TSP type then we denote the problem as p-Σ TSP using the classification scheme of Ehrgott and Gandibleux (2002). The term p-max TSP denotes a problem with p objective functions where all are BTSP type. If a problem has p_1 TSP type and p_2 BTSP type objective functions, it is denoted as p_1-Σ p_2-max TSP. In the remainder of this section, we discuss p-Σ TSP.

The distance matrix for the qth objective is $d_q, q = 1, 2, \ldots, p$ and $d_q\,[i, j]$ is the length of arc (i,j) in qth objective function. Then p-Σ TSP is:

$$\min_{\varphi \in F} d\,(\varphi) = \left(d_1\,(\varphi)\,, d_2\,(\varphi)\,, \ldots, d_p\,(\varphi)\right)$$

where $d_q(\varphi) = \sum_{i=1}^{n} d_q\,[i, \varphi\,(i)]$ and F is the set of all Hamiltonian tours.

Note that the minimization of a vector is not a well defined mathematical operation.

In multiobjective optimization problems, we may differentiate between different spaces. Two of these spaces are the decision space and the objective space. The feasible region of the decision variables is the feasible decision space. The objective space is the image of the decision space in terms of the objective function values. Let $\varphi^1, \varphi^2 \in F$ and $d\left(\varphi^i\right) = \left(d_1\left(\varphi^i\right), d_2\left(\varphi^i\right), \ldots, d_p\left(\varphi^i\right)\right)$ $i = 1, 2$. Tours φ^1 and φ^2 are feasible solutions in the decision space. Points $d\left(\varphi^1\right)$and $d\left(\varphi^2\right)$ are their images in the objective space.

Point $d\left(\varphi^1\right)$ is said to dominate point $d\left(\varphi^2\right)$ if and only if $d_q\left(\varphi^1\right) \leq d_q\left(\varphi^2\right)$ for all q and $d_q\left(\varphi^1\right) < d_q\left(\varphi^2\right)$ for at least one q. If $d_q\left(\varphi^1\right) < d_q\left(\varphi^2\right)$ for all q then $d\left(\varphi^1\right)$ is said to strictly dominate $d\left(\varphi^2\right)$. If there exists no $\varphi \in F$ such that $d\,(\varphi)$ dominates $d\left(\varphi^1\right)$, then $d\left(\varphi^1\right)$ is said to be nondominated. A point $d\left(\varphi^1\right)$ is said to be weakly nondominated if and only if there exists no point $d\,(\varphi)$, such that $d_q\left(\varphi^1\right) > d_q\,(\varphi)$ for all q. The set of weakly nondominated points includes all nondominated points and some special dominated points.

We define Y_{ND} as the set of nondominated points. For the sake of simplicity in the notation, let $y = d\,(\varphi)$ where $\varphi \in F$. Let $y \in Y_{\mathrm{ND}}$ and y^{conv} be a convex combination of the nondominated points except y. That is;

$$y^{\mathrm{conv}} = \sum_{y^k \in Y_{\mathrm{ND}} \setminus \{y\}} w^k y^k, \sum_{y^k \in Y_{\mathrm{ND}} \setminus \{y\}} w^k = 1 \text{ and } w^k \geq 0 \text{ for } y^k \in Y_{\mathrm{ND}} \setminus \{y\}\,.$$

Using the above definitions, we define three types of nondominated points. A point $y \in Y_{\mathrm{ND}}$ is said to be

- an extreme supported nondominated point if and only if there exists no y^{conv} such that $y^{\mathrm{conv}} \leq y$,
- a nonextreme supported nondominated point if and only if there exists a y^{conv} such that $y^{\mathrm{conv}} = y$,

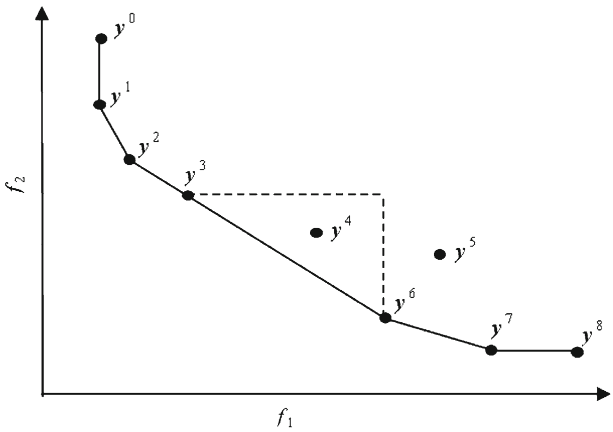

Fig. 2 Different type of points in objective space

- an unsupported nondominated point if and only if there exists a y^{conv} such that $y^{\text{conv}} \leq y$ and $y^{\text{conv}} \neq y$ (that is y^{conv} dominates y).

In Fig. 2, points y^1, y^2, y^6, and y^7 are extreme supported nondominated points, y^3 is a nonextreme supported nondominated point, and y^4 is an unsupported nondominated point. Points y^0 and y^8 are weakly nondominated but dominated. Point y^5 is strictly dominated. We refer to Steuer (1986) for a detailed coverage of various concepts and methods of multiple criteria optimization.

The terms dominance and efficiency are counterparts of each other in the objective and decision spaces, respectively. A tour $\varphi \in F$ is said to be efficient if and only if the point $d(\varphi)$ is nondominated. The tour is inefficient if and only if the point $d(\varphi)$ is dominated and the tour is weakly efficient if and only $d(\varphi)$ is weakly nondominated. Similarly, we can define extreme supported efficient, nonextreme supported efficient and unsupported efficient tours.

We will discuss our results mainly in the objective space in the rest of the paper. We should note that more than one efficient tour in the decision space may correspond to the same nondominated point in the objective space. In such cases, it is sufficient for our purposes to find one of those efficient solutions.

In multiobjective problems, nondominated points are important. The ability to find nondominated points is an important challenge in multiobjective combinatorial problems, many of which are *NP*-Hard. We first present some properties of the nondominated points for p-Σ TSP having distance matrices in $\mathbf{D}_{\text{TI}}$. We then address finding nondominated points for p-Σ TSP.

Theorem 1 *If all distance matrices are in the same class of* $\mathbf{D}_{\text{TI}}$*, then for each non-pyramidal tour there exists at least one pyramidal tour that is at least as good in every objective and possibly better in some objectives.*

Proof Since we assume that all distance matrices are in the same class of $\mathbf{D}_{\text{TI}}$, any feasible transformation does not increase the tour length in any of the objectives. In the worst case, TI results with a pyramidal tour having equal lengths in all objectives to those of the initial tour. If any of the feasible transformations used in any of the objectives is positive, then the resulting pyramidal tour dominates the initial tour. □

Corollary 1 *If all distance matrices are in the same class of* $\mathbf{D}_{\mathrm{TI}}$*, there exists a pyramidal tour corresponding to each nondominated point.*

Proof Follows directly from Theorem 1. □

Corollary 2 *If all distance matrices are in the same class of* $\mathbf{D}_{\mathrm{TI}}$*, and all feasible transformations in all distance matrices are improving, then each non-pyramidal tour is strictly dominated by at least one pyramidal tour.*

Proof Follows directly from Theorem 1. □

Example 1 The following Van der Veen matrix, $\mathbf{D}_{\mathrm{VDV}} \subseteq \mathbf{D}_{\mathrm{TI}}$ shows that there may be pyramidal tours strictly better than any other tour. The pyramidal tour $\rho = (1, 2, 3, 4)$ is shorter than any other tour for the following matrix. Therefore, no tour can be as good as ρ for this objective.

$$d = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 4 & 7 \\ 2 & 4 & 0 & 3 \\ 3 & 7 & 3 & 0 \end{bmatrix} \in \mathbf{D}_{\mathrm{VDV}} \subseteq \mathbf{D}_{\mathrm{TI}}$$

Remark 1 If distance matrices belong to different classes in $\mathbf{D}_{\mathrm{TI}}$, then a non-pyramidal tour may correspond to a unique nondominated point.

Consider the following $\mathbf{D}_{\mathrm{TI}}$ matrices (Van der Veen 1994) where $d_1 \in \mathbf{D}_{\mathrm{VDV}}$ and $d_2 \in \mathbf{D}_{\mathrm{DEMI}}$ (Demidenko matrix).

$$d_1 = \begin{bmatrix} 0 & 4 & 2 & 4 \\ 4 & 0 & 0 & 1 \\ 2 & 0 & 0 & 0 \\ 4 & 1 & 0 & 0 \end{bmatrix} \in \mathbf{D}_{\mathrm{VDV}} \quad \text{and} \quad d_2 = \begin{bmatrix} 0 & 4 & 4 & 2 \\ 4 & 0 & 1 & 0 \\ 4 & 1 & 0 & 0 \\ 2 & 0 & 0 & 0 \end{bmatrix} \in \mathbf{D}_{\mathrm{DEMI}}$$

$\rho^1 = (1, 2, 3, 4)$, $\rho^2 = (1, 2, 4, 3)$ and $\varphi = (1, 4, 2, 3)$ are all possible tours where the set of pyramidal tours is $F_{\mathrm{PYR}} = \{\rho^1, \rho^2\}$. The tour lengths are $d(\rho^1) = (8, 7)$, $d(\rho^2) = (7, 8)$ and $d(\varphi) = (7, 7)$, respectively. Both $d(\rho^1)$ and $d(\rho^2)$ are dominated by $d(\varphi)$, hence $d(\varphi)$ is the only nondominated point for these two matrices.

Let $d(\rho^q) = (d_1(\rho^q), \ldots, d_p(\rho^q))$ be the point corresponding to the shortest pyramidal tour ρ^q with respect to qth distance matrix, i.e, $d_q(\rho^q) \leq d_q(\rho)$ for any $\rho \in F_{\mathrm{PYR}}$.

Theorem 2 *If distance matrices belong to different classes in* $\mathbf{D}_{\mathrm{TI}}$*, then* $d(\rho^q)$ *is weakly nondominated.*

Proof Since ρ^q is the shortest tour in objective q, $d(\rho^q)$ is weakly nondominated. However, $d(\rho^q)$ may be dominated as demonstrated in the above remark. □

For p-Σ TSP, finding nondominated points is an important problem. We may wish to find all nondominated points or a subset that could be of interest to the decision maker.

Let us define a convex combination of p matrices that are in the same class of $\mathbf{D}_{\mathrm{TI}}$ as:

$$d_\mu = \sum_{q=1}^{p} \mu_q \cdot d_q, \sum_{q=1}^{p} \mu_q = 1 \quad \text{and} \quad \mu_q \geq 0 \; q = 1, \ldots, p.$$

In the following theorem we prove that this matrix is also in the same class of $\mathbf{D}_{\mathrm{TI}}$.

Theorem 3 *If all distance matrices are in the same class of* $\mathbf{D}_{\mathrm{TI}}$, *then any convex combination of these matrices is also in the same class of* $\mathbf{D}_{\mathrm{TI}}$.

Proof Currently existing classes of $\mathbf{D}_{\mathrm{TI}}$ are defined with inequalities like

$$\begin{aligned} &d_q[i_1, i_2] + d_q[i_3, i_4] \le d_q[i_5, i_6] + d_q[i_7, i_8] \text{ for some } q \text{ and } i_1, \ldots, i_8 \text{ or,} \\ &d_q[i_1, i_2] + d_q[i_3, i_4] + d_q[i_5, i_6] \le d_q[i_7, i_8] + d_q[i_9, i_{10}] + d_q[i_{11}, i_{12}] \\ &\qquad\qquad \text{for some } q \text{ and } i_1, \ldots, i_{12}. \end{aligned}$$

Since all convex combinations of these types of inequalities hold for all possible $i_1, \ldots, i_{12}$ used in the class definition, we conclude that all convex combinations are also in the same class. □

By convex combination of matrices, p-Σ TSP becomes a single objective TSP. For a given μ vector, an extreme supported nondominated point can be found using the dynamic program (DP) given by Gilmore et al. (1985). We refer to this DP as $\mathrm{DP}_{\mathrm{GLS}}(d)$ where d is distance matrix. All extreme supported nondominated points can be found by choosing suitableμ vectors. In the next section, we show a method for determining suitable μ for 2-Σ TSP.

Let us define the following two problems, (P_k) and $\left(P_k^{\varepsilon}\right)$ for kth objective function. Let $d_k^{\varepsilon} = d_k + \varepsilon \sum_{q=1, q \ne k}^{p} d_q$ be a distance matrix. Given a set of upper bounds by $B_q \quad q \ne k$, (P_k) finds the solution with shortest possible tour length in objective k. On the other hand, $\left(P_k^{\varepsilon}\right)$ finds a tour corresponding to a nondominated point and satisfying constraints (1) and (2).

$$(P_k) \min \sum_{i=1}^{n} d_k[i, \varphi(i)]$$

$$\text{st} \quad d_q(\varphi) = \sum_{i=1}^{n} d_q[i, \varphi(i)] \le B_q \quad \forall q \ne k \tag{1}$$

$$\varphi \in F \tag{2}$$

where F is the set of all Hamiltonian tours and B_q is an upper bound for objective q. Let $B = \left(B_1, \ldots, B_{k-1}, B_{k+1}, \ldots, B_p\right)$ be the vector of upper bounds, B_q for criteria $q \ne p$.

$$\left(P_k^{\varepsilon}\right) \quad \min \quad \sum_{i=1}^{n} d_k^{\varepsilon}[i, \varphi(i)]$$
$$\text{st} \quad (1) \text{ and } (2)$$

Let y_k^* be the optimal solution of problem (P_k) and $y^* = \left(y_1^*, \ldots, y_p^*\right)$ be the vector of tour lengths for the optimal tour of problem (P_k). Similarly, for problem $\left(P_k^{\varepsilon}\right)$, let $y_k^{*\varepsilon}$ and $y^{*\varepsilon} = \left(y_1^{*\varepsilon}, \ldots, y_p^{*\varepsilon}\right)$ be the optimal solution and optimal tour length vector, respectively. Note that, different solutions can be obtained by changing ε in $\left(P_k^{\varepsilon}\right)$. The ε value should be positive to avoid dominated points but small enough to ensure $y_k^{*\varepsilon} = y_k^*$. Steuer (1986) showed an appropriate interval for the ε value for the augmented weighted Tchebycheff program. For our case, the appropriate value of ε can be determined through Theorem 4.

Theorem 4 *Let* $y^* = \left(y_1^*, \ldots, y_p^*\right)$, $y^{*\varepsilon} = \left(y_1^{*\varepsilon}, \ldots, y_p^{*\varepsilon}\right)$ *be the optimal solutions to* (P_k) *and* $\left(P_k^\varepsilon\right)$, *respectively, and* $y = \left(y_1, \ldots, y_p\right)$*be any nondominated point satisfying* (1) *and* (2). *Then for any* $\varepsilon \in \left(0, \left(\max_{y\in S} \sum_{q\neq k} \left[y_q^{*\varepsilon} - y_q\right]\right)^{-1}\right)$ *we have* $y_k^{*\varepsilon} = y_k^*$ *where* $S = \left\{(1) \cap (2) \cap \left(y_k > y_k^{*\varepsilon}\right)\right\}$.

Proof The point $y^{*\varepsilon}$is nondominated for any$\varepsilon > 0$ due to the weighted sum objective function of problem $\left(P_k^\varepsilon\right)$.

As ε value increases, the relative importance of kth objective decreases. Increasing the ε value does not improve $y_k^{*\varepsilon}$. Then, ε should be small enough to ensure $y_k^{*\varepsilon} = y_k^*$. This implies,

$$y_k^{*\varepsilon} + \varepsilon \sum_{q\neq k} y_q^{*\varepsilon} < y_k + \varepsilon \sum_{q\neq k} y_q \tag{3}$$

for all nondominated points satisfying $y_k > y_k^*$, (1) and (2).

Since (3) needs to hold for all nondominated points satisfying $y_k > y_k^*$, $y_k^{*\varepsilon}$ cannot be greater than y_k^*, thus$y_k^* \geq y_k^{*\varepsilon}$. It is impossible to have $y_k^* > y_k^{*\varepsilon}$. Hence, $y_k^{*\varepsilon} = y_k^*$. If we can find a suitable ε satisfying inequality (3), we can rewrite it as

$$\varepsilon \sum_{q\neq k} \left[y_q^{*\varepsilon} - y_q\right] < y_k - y_k^{*\varepsilon}$$

We have two cases:

(i) If $\sum_{q\neq k} \left[y_q^{*\varepsilon} - y_q\right] > 0$ then $\varepsilon < \frac{y_k - y_k^{*\varepsilon}}{\sum_{q\neq k}\left[y_q^{*\varepsilon} - y_q\right]}$.
Since the minimum possible value for $y_k - y_k^{*\varepsilon} = 1$ (as a result of the integer edge lengths assumption), we need

$$\varepsilon < \frac{1}{\sum_{q\neq k}\left[y_q^{*\varepsilon} - y_q\right]}.$$

A general bound over all nondominated points is then

$$\varepsilon < \frac{1}{\max_{y\in S}\left\{\sum_{q\neq k}\left[y_q^{*\varepsilon} - y_q\right]\right\}}$$

where $S = (1) \cap (2) \cap \left(y_k > y_k^{*\varepsilon}\right) \cap \left(\sum_{q\neq k}\left[y_q^{*\varepsilon} - y_q\right] > 0\right)$

(ii) If $\sum_{q\neq k} \left[y_q^{*\varepsilon} - y_q\right] \leq 0$ then setting $\varepsilon \geq 0$ is sufficient.
Since $y_k^{*\varepsilon} < y_k$ and $\sum_{q\neq k} y_q^{*\varepsilon} \leq \sum_{q\neq k} y_q$, we have

$$y_k^{*\varepsilon} + \varepsilon \sum_{q\neq k} y_q^{*\varepsilon} < y_k + \varepsilon \sum_{q\neq k} y_q \quad \text{for any} \quad \varepsilon \geq 0.$$

The range is defined as:

$$\varepsilon \in \left(0, \left(\max_{y\in S} \sum_{q\neq k} \left[y_q^{*\varepsilon} - y_q\right]\right)^{-1}\right) where\ S = \left\{(1) \cap (2) \cap \left(y_k > y_k^{*\varepsilon}\right)\right\}$$

□

The above theorem gives the upper bound $\gamma_k = \left(\max_{y\in S} \sum_{q\neq k} \left[y_q^{*\varepsilon} - y_q\right]\right)^{-1}$ for ε for objective k. By taking the minimum γ_k, we generalize the upper bound for all objectives as follows.

Corollary 3 *Replacing the range of* ε *with* $\varepsilon \in \left(0, \min_k \gamma_k\right)$ *in Theorem 4, the theorem is generalized for any number of objective functions.*

To determine the above range, we need to know the set of nondominated points. This set may not be readily available, but this is not a problem in practice. A trivial upper bound for ε can be obtained by finding the total length of the longest n arcs, say UB_q, and substituting y_q for y_q^q, the shortest tour length in objective q, using $\sum_q \left[\mathrm{UB}_q - y_q^q\right]$. Then the range $\varepsilon \in \left(0, \left(\sum_q \left[\mathrm{UB}_q - y_q^q\right]\right)^{-1}\right)$ is a practical and valid range.

We develop a dynamic program to find the optimal pyramidal tour that solves $\left(P_k^\varepsilon\right)$. We define a state variable vector $R = \left(R_1, \ldots, R_{k-1}, R_{k+1}, \ldots, R_p\right)$. The number R_q corresponds to the remainder or the unconsumed portion of bound B_q by the partial tour constructed so far. Initially $R = B$ and as DP moves to inner stages, the components of R decrease.

The DP we developed is quite different than $\mathrm{DP_{GLS}}$. Given a distance matrix, $\mathrm{DP_{GLS}}$ finds the shortest pyramidal tour considering a single objective function. On the other hand, our DP considers multiple objective functions by imposing upper bounds to all but one objective. The DP either finds the shortest pyramidal tour that does not violate the upper bounds or reports that no such tour exists.

Let $C(i, j, R)$ be the length of the shortest Hamiltonian path with respect to the kth distance matrix from i to j on cities 1, 2,…,max$\{i, j\}$ that visits a subset of these nodes in a descending order from i to 1 and the remaining nodes in ascending order from 1 to j without violating the bounds, R. The state $C(i, j, R)$ finds the shortest pyramidal path in criterion k from i to j while the bounds R_q for $q \neq k$, are not violated. Let M be a sufficiently large number, i.e., $M > n.\max_{i,j}\{d_k[i, j]\}$. At state $C(i, j, R)$, there are five possible cases. If any of the upper bounds is violated then the corresponding component of R vector is negative. In this case (that corresponds to Case 1), DP returns M value for the current state. If $|i - j| > 1$ and the bounds are not violated then we consider Cases 2 or 4. In both cases, since the arc to be added is unique for each case, the selection of the next state to keep the path pyramidal is straight forward. In Case 2, arc $(j - 1, j)$ is added to the path. In Case 4, arc $(i, i - 1)$ is added to the path. In Cases 3 and 5, $|i - j| = 1$ and the bounds are not violated. In these cases, the selection of the next state is not straight forward. The minimum valued state is selected among the possible states. In all cases except for the first one, the remaining bounds are updated according to the selection of the next state.

$$C(i,j,R) = \begin{Bmatrix} 1) M & \text{if } R_q < 0 \text{ for any } q \\ 2) C(i, j-1, R - d[j-1,j]) + d_k^{\varepsilon}[j-1,j] & \text{for } i < j-1 \text{ and } R_q \geq 0\ \forall q \neq k \\ 3) \min\limits_{l<i} \left\{ C(i,l,R-d[l,j]) + d_k^{\varepsilon}[l,j] \right\} & \text{for } i = j-1 \text{ and } R_q \geq 0\ \forall q \neq k \\ 4) C(i-1, j, R - d[i,i-1]) + d_k^{\varepsilon}[i,i-1] & \text{for } i > j+1 \text{ and } R_q \geq 0\ \forall q \neq k \\ 5) \min\limits_{l<j} \left\{ C(l,j,R-d[i,l]) + d_k^{\varepsilon}[i,l] \right\} & \text{for } i = j+1 \text{ and } R_q \geq 0\ \forall q \neq k \end{Bmatrix}$$

Number of states in this DP is $O\left(n^2 \prod_{\substack{q=1 \\ q \neq k}}^{Q} B_q\right)$. The number of states is a function of the magnitudes of the upper bounds. Hence, this DP has pseudo-polynomial complexity. The optimal objective function value to $\left(P_k^{\varepsilon}\right)$ is given by

$$\text{DP}\left(d_k^{\varepsilon}, B, d\right) = \min\left\{C(n-1, n, B - d[n, n-1]) + d_k^{\varepsilon}[n, n-1], \right. \\ \left. C(n, n-1, B - d[n-1, n]) + d_k^{\varepsilon}[n-1, n]\right\}.$$

Note that in $\text{DP}\left(d_{\text{k}}^{\varepsilon}, B, d\right)$, d_k^{ε} is a distance matrix, B is a vector of upper bounds, d is a vector of distance matrices, and $d[i, j]$ is the vector of arc lengths. If there is no feasible solution for the given bounds, then the DP will return an objective function value of at least M. This DP finds the shortest pyramidal tour for any distance matrix.

The ranges developed for ε value in Theorem 4 and Corollary 3 are valid in general. If all distance matrices are in the same class of $\mathbf{D}_{\text{TI}}$, as stated in Theorem 1, then the optimal tour to $\left(P_k^{\varepsilon}\right)$ is obtained.

All nondominated points can be found by this DP by changing the B_q values. However, changing the B_q values systematically for three or more objectives is a complicated research issue that would warrant further research. In the next section, we propose an approach for finding all nondominated points for the biobjective TSP, 2-Σ TSP.

The above DP can be used for both symmetric and asymmetric matrices. If all distance matrices are symmetric then DP can be simplified as follows:

$$C(i,j,R) = \begin{Bmatrix} M & \text{if } R_q < 0 \text{ for any } q \\ C\left(i', j'-1, R - d\left[j'-1, j'\right]\right) + d_k^{\varepsilon}\left[j'-1, j'\right] & \text{for } i' < j'-1 \text{ and } R_q \geq 0\ \forall q \neq k \\ \min\limits_{l<i'} \left\{ C\left(i', l, R - d\left[l, j'\right]\right) + d_k^{\varepsilon}\left[l, j'\right] \right\} & \text{for } i' = j'-1 \text{ and } R_q \geq 0\ \forall q \neq k \end{Bmatrix}$$

where $i' = \min(i, j)$ and $j' = \max(i, j)$. In this case, the optimal objective function value also simplifies to

$$\text{DP}\left(d_k^{\varepsilon}, B, d\right) = C(n-1, n, B - d[n, n-1]) + d_k^{\varepsilon}[n, n-1].$$

4 The biobjective TSP

We develop an approach to generate all nondominated points for 2-Σ TSP. We first find the extreme supported nondominated points by using the weighting scheme proposed by Aneja and Nair (1979). Then we search for the nonextreme supported nondominated and unsupported nondominated points between each adjacent pair of extreme supported nondominated points.

We define nonextreme supported nondominated and unsupported nondominated points as nonextreme nondominated points, because we do not need to differentiate between these two types of points in our method. Let Y_{E} and Y_{NE} be the sets of extreme supported nondominated points and nonextreme nondominated points, respectively.

Consider the optimal objective function values of the single objective TSPs, $y_q^q = \min_{\varphi \in F} d_q(\varphi)$, and let φ^q be the corresponding optimal tours for $q = 1, 2$. Let $y_2^1 = d_2(\varphi^1)$, $y_1^2 = d_1(\varphi^2)$, $y^1 = (y_1^1, y_2^1)$ and $y^2 = (y_1^2, y_2^2)$. Without loss of generality, assume that $y_1^1 < y_1^2$ and $y_2^1 > y_2^2$. If $y^1 = y^2$ or $y^1 \leq y^2$ or $y^2 \leq y^1$ then there is a unique nondominated point and the problem is trivial.

Theorem 5 *Using ε in the range $\left(0, \min\left\{\frac{1}{y_2^1 - y_2^2}, \frac{1}{y_1^2 - y_1^1}\right\}\right)$ is sufficient to avoid nondominated points in problem $\left(P_k^{\varepsilon}\right)$.*

Proof Follows directly from Theorem 4. □

Note that both points y^1 and y^2 can be weakly nondominated. Let $y^{q-\mathrm{ND}}$ be a nondominated point satisfying $y_q^q = y_q^{q-\mathrm{ND}}$ for $q = 1,2$. These nondominated points can be determined by the DP we developed. Using these points a larger upper bound for ε can be obtained.

Corollary 4 *The upper bound for ε can be replaced by the following term:*

$$\min\left\{\frac{1}{y_2^{1-ND} - y_2^2}, \frac{1}{y_1^{2-ND} - y_1^1}\right\}.$$

Proof The points y^1 and y^2 may be weakly nondominated. An overestimated range (for nondominated points) is obtained by the denominator term using y^1 and y^2. If nondominated points are used in the denominator, the range (for nondominated points) may decrease and the upper bound value for ε may increase. □

We define two algorithms to find all points in Y_{E}; Recursive (y^a, y^b) and *A1*. Recursive (y^a, y^b) finds all extreme supported nondominated points between two given extreme supported nondominated points y^a and y^b. *A1* solves (P_1^{ε}) and (P_2^{ε}). If two different solutions are obtained in *A1* then Recursive (y^a, y^b) is called. $\mathrm{DP_{GLS}}(d)$ finds the shortest pyramidal tour with respect to distance matrix d and returns the point $y = (y_1, y_2)$

We can obtain the extreme supported nondominated points in $O(n^2)$ using distance matrices d_1^{ε} and d_2^{ε}. The extreme supported nondominated points can be determined by changing the weight μ of matrix $d^{\mu} = \mu d_1 + (1-\mu) d_2$, $\mu \in (0, 1)$ and applying $\mathrm{DP_{GLS}}(d^{\mu})$ for the resulting single objective problem. For each weight set, a solution is obtained in $O(n^2)$.

A1

Initialization: Set $Y_E = \emptyset$.

Step 1. Solve $\mathrm{DP_{GLS}}(d_1^{\varepsilon})$, let the optimal point be y^1.
Step 2. Solve $\mathrm{DP_{GLS}}(d_2^{\varepsilon})$, let the optimal point be y^2.
Step 3. If $y_1^1 < y_1^2$ and $y_2^2 < y_2^1$ then go to Step 5 else go to Step 4.
Step 4. If $y^1 = y^2$ then single optimal solution is y^1, $Y_E = \{y^1\}$, go to Step 7.
Step 5. $Y_E = \{y^1, y^2\}$.
Step 6. Call Recursive (y^1, y^2).
Step 7. Terminate the algorithm.

Recursive$\left(y^a, y^b\right)$

Step 1. Set $\mu = \frac{y_2^a - y_2^b}{(y_1^a - y_2^a) - (y_1^b - y_2^b)}$.
Step 2. Set $d^\mu = \mu d_1 + (1 - \mu) d_2$.
Step 3. Solve $\mathrm{DP_{GLS}}$ (d^μ), let the solution be y^{new}.
Step 4. If $y^{\text{new}} \notin Y_E$ then go to Step 5
else $Y_E = Y_E \cup \{y^{\text{new}}\}$,
Call Recursive (y^a, y^{new}),
Call Recursive $\left(y^{\text{new}}, y^b\right)$.
Step 5. Terminate the algorithm.

If $|Y_\mathrm{E}| \geq 2$ then we search for the nonextreme nondominated points using the DP we developed. Since there are only two objectives, we use the state variable $C\,(i, j, R)$ where R is a scalar. For each consecutive extreme supported nondominated point pair in Y_E, nonextreme nondominated points should be searched between them. A point is obtained in $O(n^2 B)$ where B is the upper bound on one of the objectives. Without loss of generality, we select to use the second objective as a bound and minimize the augmented version of the first objective. Algorithm *A2* is used to find all nonextreme nondominated points in Y_NE. We refer to our DP as $DP\left(d_1^\varepsilon, B_2, d_2\right)$ where d_1^ε and d_2 are distance matrices and B_2 is a scalar.

The set of nondominated points is $Y_\mathrm{ND} = Y_\mathrm{E} \cup Y_\mathrm{NE}$. We can find a nondominated point satisfying the given bound with a pseudo-polynomial DP. However, the complexity of identifying all nondominated points is still an open problem.

A2
Initialization: Set $Y_\mathrm{NE} = \emptyset$.
Sort elements of Y_E, such that $y_1^{[1]} < y_1^{[2]} < \ldots < y_1^{[|Y_\mathrm{E}|]}$.
Set $r = 1$.

Step 1. Solve $\mathrm{DP2}\left(d_1^\varepsilon, y_2^{[r]} - 1, d_2\right)$, let the resulting point be $y = (y_1, y_2)$.
Step 2. If $y = y^{[r+1]}$ then r=r+1.
If $r = |Y_\mathrm{E}|$ then go to Step 6 else go to Step 1.
Step 3. $Y_\mathrm{NE} = Y_\mathrm{NE} \cup \{y\}$.
Step 4. If $y_2 - 1 = y_2^{[r+1]}$ then r=r+1, go to Step 1.
Step 5. Solve $\left(d_1^\varepsilon, y_2^{[r]} - 1, d_2\right)$, let the resulting point be $y = (y_1, y_2)$, go to Step 2.
Step 6. Terminate the algorithm.

5 The bottleneck TSP

The optimal pyramidal tour for the Bottleneck TSP can be found in $O(n^2)$ with a small modification in $\mathrm{DP_{GLS}}$ (d). Burkard and Sandholzer (1991) studied the polynomially solvable special cases of the Bottleneck TSP. They presented several conditions for pyramidally solvable Bottleneck TSPs.

One may be curious to know whether some results on pyramidal tours are applicable to the bottleneck-type objectives. For some classes in $\mathbf{D}_\mathrm{PYR}$, using the "maximum" operator instead of the "sum" operator in the distance matrix definition results in a class which is also in $\mathbf{D}_\mathrm{PYR}$. The class of Monge matrices is such an example (Burkard and Sandholzer 1991).

In a similar way, we can define the bottleneck version of the Van der Veen matrix as follows:

$$\mathbf{D}_{\mathrm{BVDV}} = \left\{ d[i,j] \left| \begin{array}{ll} d[i,j] = d[j,i] & \text{for all } i \text{ and } j \\ \max\{d[i,j], d[j+1,k]\} & \\ \leq \max\{d[i,k], d[j,j+1]\} & \text{for all } i < j < j+1 < k \end{array} \right. \right\}.$$

Theorem 6 $\mathbf{D}_{\mathrm{BVDV}} \notin \mathbf{D}_{\mathrm{PYR}}$.

Proof We provide a counter example. Note that $d \in \mathbf{D}_{\mathrm{BVDV}}$ for $y \geq 1$. For the distance matrix given below, the length of tour $\rho = (1, 4, 2, 6, 3, 5)$ is 1. However, all pyramidal tours have tour lengths of y.

$$d = \begin{bmatrix} - & y & 0 & 0 & 1 & y \\ y & - & y & 0 & 0 & 1 \\ 0 & y & - & y & 1 & 1 \\ 0 & 0 & y & - & y & 0 \\ 1 & 0 & 1 & y & - & 0 \\ y & 1 & 1 & 0 & 0 & - \end{bmatrix} \in \mathbf{D}_{\mathrm{BVDV}} \notin \mathbf{D}_{\mathrm{PYR}}$$

Increasing the value of y in $d \in \mathbf{D}_{\mathrm{BVDV}}$, the lengths of pyramidal tours can be increased arbitrarily. □

We next define the bottleneck version of the Demidenko matrix as follows:

$$\mathbf{D}_{\mathrm{BDEMI}} = \left\{ d[i,j] \left| \begin{array}{ll} d[i,j] = d[j,i] & \text{for all } i \text{ and } j \\ \max\{d[i,j], d[j+1,k]\} & \\ \leq \max\{d[i,j+1], d[j,k]\} & \text{for all } i < j < j+1 < k \end{array} \right. \right\}.$$

Theorem 7 $\mathbf{D}_{\mathrm{BDEMI}} \notin \mathbf{D}_{\mathrm{PYR}}$.

Proof We provide a counter example. Note that $d \in \mathbf{D}_{\mathrm{BDEMI}}$ for $y \geq 0$. For the distance matrix given below, the length of tour $\rho = (1, 5, 3, 4, 2, 6)$ is 0. However, all pyramidal tours have tour lengths of y.

$$d = \begin{bmatrix} - & y & y & y & 0 & 0 \\ y & - & 0 & 0 & 0 & 0 \\ y & 0 & - & 0 & 0 & y \\ y & 0 & 0 & - & 0 & y \\ 0 & 0 & 0 & 0 & - & 0 \\ 0 & 0 & y & y & 0 & - \end{bmatrix} \in \mathbf{D}_{\mathrm{BDEMI}} \notin \mathbf{D}_{\mathrm{PYR}}$$

Increasing the value of y in $d \in \mathbf{D}_{\mathrm{BDEMI}}$ the lengths of pyramidal tours can be increased arbitrarily. □

Since $\mathbf{D}_{\mathrm{BVDV}}$ and $\mathbf{D}_{\mathrm{BDEMI}}$ do not guarantee that the optimal tour is pyramidal, we consider the bottleneck type objectives no further.

6 Conclusions

In this study, we studied the multiobjective TSP on $\mathbf{D}_{\mathrm{TI}} \in \mathbf{D}_{\mathrm{PYR}}$ and showed some properties of nondominated points. We developed a pseudo-polynomial DP to find a nondominated point to the problem when all distance matrices are in the same class of $\mathbf{D}_{\mathrm{TI}}$. For the biobjective case, we developed an approach to find all nondominated points. We also demonstrated that bottleneck types of Van der Veen matrices and Demidenko matrices are not in $\mathbf{D}_{\text{PYR}}$, and hence the developments are not applicable to these cases.

A further research direction is the extension of these results to other polynomially solvable cases of the TSP and other combinatorial optimization problem. Also, developing algorithms for finding all nondominated points for 3 or more objectives is a challenging and interesting future research area.

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